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Symmetry group for a completely symmetric vertex model†

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Abstract. The symmetry group for the partition function of the completely X symmetric vertex model is determined. Here X is an arbitrary finite Abelian group.

1. Introduction

Let Λ be a square lattice with M rows and N columns. We assume periodic boundary conditions. Let X be any finite Abelian group. A configuration ω is defined by assigning elements of X to the bonds of Λ; let Ω_Λ be the set of all such configurations ω. To each vertex with bond states $x, y, u, v \in X$ is assigned a Boltzmann weight S_{xy}^{uv}. The weight of a configuration $\omega \in \Omega_\Lambda$ is defined to be the product over all lattice sites in Λ of the individual Boltzmann weights. Thus the partition function Z_Λ is given by

$$Z_\Lambda = \sum_{\omega \in \Omega_\Lambda} \prod S_{xy}^{uv}. \tag{1.1}$$

Following Belavin (1981) and Chudnovsky and Chudnovsky (1981), we say such a vertex model is completely X symmetric if the Boltzmann weights satisfy

$$S_{xy}^{uv} = 0 \qquad \text{unless } x + y = u + v \tag{1.2a}$$

and

$$S_{x+z\, y+z}^{u+z\, v+z} = S_{x\,y}^{u\,y} \qquad \text{for every } x, y, u, v, z \in X \tag{1.2b}$$

where we have written the group law of X in additive notation. For the case $X = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ the above vertex model is Baxter's symmetric eight-vertex model (Baxter 1972, 1982). If $|X|$ denotes the order of X, then this X symmetric model depends upon $|X|^2$ independent parameters (one of these being an overall normalisation). We write $Z_\Lambda[S]$ to denote the dependence of Z_Λ upon the Boltzmann weights S_{xy}^{uv}.

For $X = \mathbb{Z}_2$ Fan and Wu (1970) (see also Fan 1972, Johnson *et al* 1973) discovered that Z_Λ has certain symmetries. To state their results, we write S as

$$S = \sum_{j=0}^{3} w_j \sigma^j \otimes \sigma^j \tag{1.3}$$

where σ^0 is the 2×2 identity matrix and σ^j, $j = 1, 2, 3$, are the Pauli spin matrices. Then Fan and Wu proved

$$Z_\Lambda[\pm w_{i_0}, \pm w_{i_1}, \pm w_{i_2}, \pm w_{i_3}] = Z_\Lambda[w_0, w_1, w_2, w_3] \tag{1.4}$$

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where (i_0, i_1, i_2, i_3) is any permutation of $(0, 1, 2, 3)$ and $\pm$ denotes any choice of signs. This symmetry is important in Baxter's solution of the symmetric eight-vertex model (Baxter 1972, 1982). It is the purpose of this paper to determine the symmetry group for $Z_\Lambda[S]$ when X is any finite Abelian group and S is completely X symmetric.

As was first emphasised by Belavin (1981) and Chudnovsky and Chudnovsky (1981) (see also Bovier 1983, Cherednik 1982), the Heisenberg group is important in any discussion of the X symmetric model. To establish notation and to make the paper self-contained, we collect in § 2 those results we need concerning the Heisenberg group. Our principal mathematical references are Mumford (1983) and Siegel (1971). We have included a self-contained proof of the well known result in lemma 2.1. Presumably lemma 2.2 is also well known but we could not find any explicit references. In § 3 we determine the symmetry properties of $Z_\Lambda[S]$ following the methods of Richey and Tracy (1986) where the case $X = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ was first considered. These authors in turn built on the work of Fan and Wu (1970), Fan (1972), Wegner (1971, 1973) and Johnson *et al* (1973). In particular, the idea to use the similarity transformation in the proof of theorem 3.1 can be traced back to these early papers. Here, we have shown the generality of this method and have clarified the role the Heisenberg group plays in the symmetries. In § 4 we discuss the connection between theorem 3.1 for $X = \mathbb{Z}_n$ and previous work (Richey and Tracy 1986). The results obtained here require no restrictions on the Boltzmann weights S_{ij}^{kl}. Normally restrictions on multistate vertex models arise when one demands commutativity of the transfer matrices. Further discussion of the relationship between symmetries and commutativity of transfer matrices can be found, for the case $X = \mathbb{Z}_n$, in Richey and Tracy (1986).

2. Heisenberg group for a finite Abelian group

As is well known any finite Abelian group X, with $|X| = n$, is isomorphic to

$$X \simeq \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r} \tag{2.1}$$

where $n = n_1 \ldots n_r$, n_i a prime power. Also distinct choices of the n_i result in distinct (non-isomorphic) finite Abelian groups. If $X = \mathbb{Z}_n$, then the decomposition $n = n_1 \ldots n_r$ is the prime decomposition of n with each of the primes distinct. In this case, the isomorphism $\mathbb{Z}_n \to \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r}$ is commonly referred to as the Chinese remainder theorem (CRT).

An element α' of X may be identified with an r-tuple $\alpha' = (\alpha_1', \ldots, \alpha_r')$, $\alpha_i' \in \mathbb{Z}_{n_i} = \{0, 1, \ldots, n_i - 1\}$. Let $\mathbb{Z}_{n_i}^2 = \mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i}$ and $G_n = \mathbb{Z}_{n_1}^2 \times \ldots \times \mathbb{Z}_{n_r}^2$. Thus an element $\alpha \in G_n$ can be written as

$$\alpha = (\alpha', \alpha'') \qquad \alpha', \alpha'' \in X = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r}$$

or

$$\alpha = (\alpha_1, \ldots, \alpha_r) \qquad \alpha_i = (\alpha_i', \alpha_i'') \in \mathbb{Z}_{n_i}^2 \tag{2.2}$$

or

$$\alpha = (\alpha_1', \ldots, \alpha_r', \alpha_1'', \ldots, \alpha_r'') \qquad \alpha_i', \alpha_i'' \in \mathbb{Z}_{n_i}.$$

The method of expressing α will depend upon the context.

We now define a Heisenberg group associated with X. This could be defined abstractly, but we prefer to give first the 'coordinate version' as it proves useful in the statistical mechanics. For an arbitrary positive integer $n_k(>1)$ we define operators g_k and h_k by

$$\begin{aligned} g_k e_j &= \omega_k^j e_j \qquad j=0,1,\ldots,n_k-1 \\ h_k e_j &= e_{j+1} \end{aligned} \tag{2.3}$$

where $\{e_j\}_{j\in\mathbb{Z}_{n_k}}$ is the standard basis of $\mathbb{C}^{n_k}$ and $\omega_k=\exp(2\pi i/n_k)$. Let

$$I_{\alpha_k}=h_k^{\alpha'_k}g_k^{\alpha''_k} \qquad \alpha_k=(\alpha'_k,\alpha''_k)\in\mathbb{Z}^2_{n_k} \tag{2.4}$$

then

$$I_{\alpha_k}I_{\beta_k}=\omega_k^{\langle\beta_k,\alpha_k\rangle}I_{\beta_k}I_{\alpha_k} \tag{2.5}$$

with $\langle\beta_k,\alpha_k\rangle=\beta'_k\alpha''_k-\beta''_k\alpha'_k$. Let H_{n_k} be the group of operators generated by I_{α_k}, $\alpha_k\in\mathbb{Z}^2_{n_k}$. Then H_{n_k} is the Heisenberg group associated with $\mathbb{Z}_{n_k}$. For X identified as in (2.1) we define the associated Heisenberg group H_X by

$$\mathrm{H}_X=\mathrm{H}_{n_1}\otimes\ldots\otimes\mathrm{H}_{n_r}. \tag{2.6}$$

We write

$$I_\alpha=I_{\alpha_1}\otimes\ldots\otimes I_{\alpha_r} \qquad \alpha=(\alpha_1,\ldots,\alpha_r),\ \alpha_i\in\mathbb{Z}^2_{n_i} \tag{2.7}$$

and observe for all $\alpha,\beta\in X\times X$

$$I_\alpha I_\beta=\omega^{k(\beta,\alpha)}I_{\alpha+\beta}=\omega^{B(\beta,\alpha)}I_\beta I_\alpha \tag{2.8}$$

where

$$k(\beta,\alpha)=\sum_{i=1}^{r}\frac{n}{n_1}\alpha''_i\beta'_i \tag{2.9a}$$

$$\begin{aligned} B(\beta,\alpha)&=k(\beta,\alpha)-k(\alpha,\beta) \\ &=\sum_{i=1}^{r}\frac{n}{n_i}(\beta'_i\alpha''_i-\beta''_i\alpha'_i) \end{aligned} \tag{2.9b}$$

and $\omega=\exp(2\pi i/n)$.

Lemma 2.1. The action of H_X on $\mathbb{C}^n\approx\mathbb{C}^{n_1}\otimes\ldots\otimes\mathbb{C}^{n_r}$ is irreducible.

Proof. Let x be an arbitrary non-zero element of $\mathbb{C}^n$

$$x=\sum_{i=(i_1,\ldots,i_r)\in\mathbb{Z}_{n_1}\times\ldots\times\mathbb{Z}_{n_r}}c_ie_{i_1}\otimes\ldots\otimes e_{i_r}.$$

We will show that the action of H_X on x spans $\mathbb{C}^n$. If we can show that $e_0\otimes\ldots\otimes e_0$ is in the span of H_X on x, then we are done since

$$h_1^{i_1}\otimes\ldots\otimes h_r^{i_r}e_0\otimes\ldots\otimes e_0=e_{i_1}\otimes\ldots\otimes e_{i_r}.$$

Since $x\neq0$ one of the coefficients $c_i=c_{i_1\ldots i_r}\neq0$. Applying $h_1^{-i_1}\otimes\ldots\otimes h_r^{-i_r}$ to x allows us to assume that $c_0=c_{0\ldots0}\neq0$. Now if

$$\begin{aligned} x_\beta &:= g_1^{\beta''_1}\otimes\ldots\otimes g_r^{\beta''_r}x \\ &=\sum_i c_i\omega_1^{\beta''_1 i_1}\ldots\omega_r^{\beta''_r i_r}e_{i_1}\otimes\ldots\otimes e_{i_r} \end{aligned}$$

then

$$\sum_{\beta''_1,\ldots,\beta''_r}x_\beta=nc_0e_0\otimes\ldots\otimes e_0$$

since $\Sigma_{\beta''_k}\omega_k^{\beta''_k i_k}=n_k\delta_{0,i_k}$. Since $c_0\neq0$, e_0 is in the subspace spanned by H_X on x; and hence the lemma has been proved.

Let E denote the $r\times r$ diagonal matrix with diagonal entries n/n_i and define the $2r\times 2r$ matrix J by

$$J=\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}. \tag{2.10}$$

Then in obvious matrix notation we can write (2.9) as

$$k(\alpha,\beta)={}^{\mathrm{t}}\alpha' E\beta'' \tag{2.11a}$$

and

$$B(\alpha,\beta)={}^{\mathrm{t}}\alpha J\beta \tag{2.11b}$$

where $\alpha=(\alpha',\alpha'')$, $\beta=(\beta',\beta'')\in G_n$ (and are written as column vectors).

It will be convenient to introduce the Siegel upper half-space $\mathscr{H}_r$ which consists of all $r\times r$ complex symmetric matrices whose imaginary parts are positive definite. Let $\Omega\in\mathscr{H}_r$, denote by $\Omega_1,\ldots,\Omega_r$ the columns of Ω and $e_1,\ldots,e_r$ the standard basis of $\mathbb{C}^r$. Then to each $\alpha\in G_n$ we associate a point on the lattice

$$L_\Omega=\Omega\mathbb{Z}^r+E\mathbb{Z}^r \tag{2.12}$$

by

$$\alpha\mapsto\alpha_1'\Omega_1+\ldots+\alpha_r'\Omega_r+\alpha_1''(n/n_1)e_1+\ldots+\alpha_r''(n/n_r)e_r \tag{2.13a}$$

or, more compactly,

$$\alpha\mapsto\Omega\alpha'+E\alpha''. \tag{2.13b}$$

This mapping forms a natural association of elements I_α of H_X with lattice points of L_Ω. In particular, changing generators of H_X is equivalent to changing generators of L_Ω. Consider the group of matrices Γ_E defined by

$$\Gamma_E=\left\{\gamma=\begin{pmatrix} A & B \\ C & D\end{pmatrix}: A,\ B,\ C,\ D,\ r\times r \text{ matrices with integer entries and } {}^{\mathrm{t}}\gamma J\gamma=J\right\}. \tag{2.14}$$

Note for $E=I$, $\Gamma_E=\mathrm{Sp}(2r,\mathbb{Z})$, and for any $\gamma\in\Gamma_E$, $\det\gamma=\pm1$. Let

$$\begin{pmatrix} P \\ Q\end{pmatrix}=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix}\Omega \\ E\end{pmatrix}$$

where $\gamma=\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\in\Gamma_E$, then $({}^{\mathrm{t}}P,{}^{\mathrm{t}}Q)$ are new generators of L_Ω. To see this, we need only show that for any $\alpha=\Omega\alpha'+E\alpha''\in L_\Omega$, there is a $\xi=(\xi',\xi'')\in G_n$ such that

$$\begin{aligned}\alpha&={}^{\mathrm{t}}P\xi'+{}^{\mathrm{t}}Q\xi''\\ &=({}^{\mathrm{t}}\Omega{}^{\mathrm{t}}A+{}^{\mathrm{t}}E{}^{\mathrm{t}}B)\xi'+({}^{\mathrm{t}}\Omega{}^{\mathrm{t}}C+{}^{\mathrm{t}}E{}^{\mathrm{t}}D)\xi''\\ &=\Omega({}^{\mathrm{t}}A\xi'+{}^{\mathrm{t}}C\xi'')+E({}^{\mathrm{t}}B\xi'+{}^{\mathrm{t}}D\xi'').\end{aligned}$$

That is,

$$\begin{pmatrix}\alpha' \\ \alpha''\end{pmatrix}=\begin{pmatrix}{}^{\mathrm{t}}A & {}^{\mathrm{t}}C \\ {}^{\mathrm{t}}B & {}^{\mathrm{t}}D\end{pmatrix}\begin{pmatrix}\xi' \\ \xi''\end{pmatrix}={}^{\mathrm{t}}\gamma\xi.$$

Since Γ_E is a group, $\xi={}^{\mathrm{t}}(\gamma^{-1})\alpha$ is an element of G_n. For $\gamma\in\Gamma_E$ let $\gamma_1,\ldots,\gamma_r,\delta_1,\ldots,\delta_r$ denote the $2r$ columns of γ. To each γ_i, δ_i we associate an element of G_n by $\gamma_i=(\gamma_i',\gamma_i'')$

where γ_i' consists of the first r rows of γ_i and γ_i'' consists of the last r rows, and similarly for $\delta_i = (\delta_i', \delta_i'')$.

For any $\gamma \in \Gamma_E$ we define a family of Heisenberg elements

$$I_\alpha^\gamma = (I_{\gamma_1})^{\alpha_1'} \dots (I_{\gamma_r})^{\alpha_r'} (I_{\delta_1})^{\alpha_1''} \dots (I_{\delta_r})^{\alpha_r''} \qquad \alpha \in G_n. \tag{2.15}$$

It is easy to check that

$$I_\alpha^\gamma = \omega^\rho I_{\gamma\alpha} \qquad \rho \in \mathbb{Z}. \tag{2.16}$$

Then we have the following lemma.

Lemma 2.2. For each $\gamma \in \Gamma_E$ there exists an invertible matrix $U_\gamma : \mathbb{C}^n \to \mathbb{C}^n$ such that for all $\alpha \in G_n$

$$I_\alpha^\gamma = \lambda U_\gamma I_\alpha U_\gamma^{-1}$$

where λ is some scalar depending on α.

Proof. Let $\gamma \in \Gamma_E$ and denote ξ_k the $2r \times 1$ column vector with all zeros except for a 1 at the kth row.

Then for all $k, l = 1, \dots, r$

$$B(\gamma_k, \gamma_l) = B(\gamma\eta_k, \gamma\eta_l) = B(\eta_k, \eta_l) = 0$$

$$B(\delta_k, \delta_l) = B(\eta_{k+r}, \eta_{l+r}) = 0$$

$$B(\gamma_k, \delta_l) = B(\eta_k, \eta_{l+r}) = (1/n_k)\delta_{k,l}.$$

Thus

$$I_{\gamma_k} I_{\gamma_l} = I_{\gamma_l} I_{\gamma_k} \tag{2.17a}$$

$$I_{\delta_k} I_{\delta_l} = I_{\delta_l} I_{\delta_l} \tag{2.17b}$$

and

$$I_{\gamma_k} I_{\delta_l} = \begin{cases} I_{\delta_l} I_{\gamma_k} & \text{if } k \neq l \\ \omega_k^{-1} I_{\delta_l} I_{\gamma_k} & \text{if } k = l. \end{cases}$$

Now let x_0 be any common eigenvector of the I_{δ_i}, $i = 1, \dots, r$:

$$I_{\delta_i} x_0 = \lambda_i x_0.$$

Define $x_j = x_{j_1 \dots j_r}$ by

$$x_j = (I_{\gamma_1})^{j_1} \dots (I_{\gamma_r})^{j_r} x_0 \qquad j_k \in \mathbb{Z}_{n_k}.$$

Then using (2.17) we see that

$$I_{\delta_1}^{j_1'} \dots I_{\delta_r}^{j_r'} x_{j_1 \dots j_r} = \lambda_1^{j_1'} \dots \lambda_r^{j_r'} \omega^{\Sigma_{k=1}^r (n/n_k) j_k j_k'} x_{j_1 \dots j_r}$$

and

$$I_{\gamma_1}^{j_1'} \dots I_{\gamma_r}^{j_r'} x_{j_1 \dots j_r} = x_{j_1 + j_1', \dots, j_r + j_r'}.$$

Since the action of H_X on $\mathbb{C}^n$ is irreducible, we conclude that the set $\{x_{j_1 \dots j_r} : (j_1 \dots j_r) \in \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_r}\}$ is a basis for $\mathbb{C}^n$. Let U_γ be the change of basis matrix

$$U_\gamma e_j = x_j \qquad j = (j_1, \dots, j_r).$$

Then

$$\begin{aligned} U_\gamma^{-1} I_{\delta_k} U_\gamma e_j &= U_\gamma^{-1} I_{\delta_k} x_j \\ &= U_\gamma^{-1} \lambda_k \omega^{(n/n_k) j_k} x_j \\ &= \lambda_k \omega_k^{j_k} e_j \end{aligned}$$

and

$$\begin{aligned} U_\gamma^{-1} I_{\gamma_k} U_\gamma e_j &= U_\gamma^{-1} I_{\gamma_k} x_j \\ &= U_\gamma^{-1} x_{j_1, \ldots, j_k+1, \ldots, j_r} \\ &= e_{j_1 \ldots j_k+1 \ldots j_r}. \end{aligned}$$

That is, $\lambda_k^{-1} U_\gamma^{-1} I_{\delta_k} U_\gamma$ is $I \otimes \ldots \otimes g \otimes \ldots \otimes I$ (g at the kth slot) and $U_\gamma^{-1} I_{\gamma_k} U_\gamma$ is $I \otimes \ldots \otimes h \otimes \ldots \otimes I$ (h at the kth slot). Then an easy calculation shows that

$$I_\alpha^\gamma = \lambda U_\gamma I_\alpha U_\alpha^{-1} \qquad \text{where} \qquad \lambda = \lambda_1^{\alpha_1''} \ldots \lambda_r^{\alpha_r''}.$$

3. Symmetries of the partition function

The Boltzmann weights S_{ij}^{kl}, $i, j, k, l \in X$ define a completely X symmetric matrix S in the standard basis $\{e_i \otimes e_j\}_{i,j \in X}$ for $\mathbb{C}^n \times \mathbb{C}^n$. Belavin (1981) and Chudnovsky and Chudnovsky (1981) have shown that a $n^2 \times n^2$ S is completely X symmetric (satisfies conditions (1.2)) if and only if

$$S = \sum_{\alpha \in G_n} w_\alpha I_\alpha \otimes I_\alpha^{-1}. \tag{3.1}$$

The partition function, Z_Λ, for a finite rectangular lattice Λ with cyclic boundary conditions may be written as

$$Z_\Lambda = \mathrm{Tr}(T^M) \qquad M = \text{number of rows of } \Lambda$$

where

$$T_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathrm{Tr}(L(\alpha_1, \beta_1) \ldots L(\alpha_N, \beta_N))$$

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \qquad \boldsymbol{\beta} = (\beta_1, \ldots, \beta_N) \qquad \alpha_i, \beta_i \in X$$

N is the number of columns of Λ and $L(\alpha, \beta)$ is the $n \times n$ matrix defined by

$$(L(\alpha, \beta))_{\lambda,\mu} = S_{\lambda\alpha}^{\mu\beta} \qquad \alpha, \beta, \lambda, \mu \in X.$$

Due to the invariance of the trace under similarity transformations, it is easy to see that if

$$S \to U_1 \otimes U_2 S U_1^{-1} \otimes U_2^{-1} = \tilde{S}$$

where U_1 and U_2 are any $n \times n$ invertible matrices, then

$$Z_\Lambda[\tilde{S}] = Z_\Lambda[S].$$

We will now consider some special cases of similarity transformations; in particular, those associated with the Heisenberg group H_X. First choose

$$U_1 = I_\beta \qquad U_2 = I = \text{identity}$$

and consider

$$\tilde{S} = I_\beta \otimes ISI_\beta^{-1} \otimes I = I \otimes I_\beta^{-1} SI \otimes I_\beta.$$

In terms of the w_α we have using (2.8)

$$I_\beta \otimes ISI_\beta^{-1} \otimes I = \sum \omega^{B(\alpha,\beta)} w_\alpha I_\alpha \otimes I_\alpha^{-1}.$$

Hence the similarity transformation on the S matrix $S \mapsto I_\beta \otimes ISI_\beta^{-1} \otimes I$ is equivalent to the transformation on the w_α coordinates

$$w_\alpha \mapsto \omega^{B(\alpha,\beta)} w_\alpha.$$

The second type of similarity transformation we will consider arises from the choice of new generators for H_X. For $\gamma \in \Gamma_E$, we define

$$S^\gamma = \sum_{\xi \in G_n} w_\xi I_\xi^\gamma \otimes (I_\xi^\gamma)^{-1}$$

where I_ξ^γ is defined by (2.15). By lemma 2.2 we have

$$S^\gamma = U_\gamma \otimes U_\gamma S U_\gamma^{-1} \otimes U_\gamma^{-1}$$

so that

$$Z_\Lambda[S^\gamma] = Z_\Lambda[S].$$

By (2.16)

$$I_\xi^\gamma \otimes (I_\xi^\gamma)^{-1} = I_{\gamma\xi} \otimes I_{\gamma\xi}^{-1}$$

so that

$$S^\gamma = \sum w_\xi I_{\gamma\xi} \otimes I_{\gamma\xi}^{-1} = \sum w_{\gamma\xi}^{-1} I_\xi \otimes I_\xi^{-1}.$$

Hence the similarity transformation $S \mapsto S^{\gamma^{-1}}$ corresponds to the transformation $w_\xi \mapsto w_{\gamma\xi}$. We now have the following theorem.

Thereom 3.1. Let S be any X symmetric matrix, $Z_\Lambda[S]$ the corresponding partition function for a finite rectangular lattice with cyclic boundary conditions. Write

$$S = \sum_{\xi \in G_n} w_\xi I_\xi \otimes I_\xi^{-1}.$$

Then $Z_\Lambda[S]$ is invariant under the transformations on S given by

$$w_\xi \mapsto \omega^{B(\xi,\alpha)} w_\xi \qquad \alpha \in \mathbb{Z}^r \times \mathbb{Z}^r$$

$$w_\xi \mapsto w_{\gamma\xi} \qquad \gamma \in \Gamma_E.$$

The α only depends on its coset in $(\mathbb{Z}/n_1\mathbb{Z} \times \ldots \times \mathbb{Z}_{n_r}\mathbb{Z})^2$ and γ only depends on its coset in $\Gamma_E \backslash \Gamma_E(n_1, \ldots, n_r)$ where

$$\Gamma_E(n_1, \ldots, n_r) = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_E : \begin{pmatrix} A_{kk} & B_{kk} \\ C_{kk} & D_{kk} \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n_k, \text{ and } A_{ki} \equiv B_{ki} \right.$$

$$\left. \equiv C_{ki} \equiv D_{ki} \equiv 0 \bmod n_k \text{ for } i \neq k \right\}.$$

Also, the action of Γ_E normalises the action of $\mathbb{Z}^r \times \mathbb{Z}^r$ showing that the symmetry group is

$$\Gamma_E \rtimes (\mathbb{Z}^r \times \mathbb{Z}^r).$$

Proof. The invariance of the partition function has already been demonstrated. The statement that α only depends on its coset in $(\mathbb{Z}/n_1\mathbb{Z}\times\ldots\times\mathbb{Z}/n_r\mathbb{Z})^2$ follows from if $\beta\in(n_1\mathbb{Z}\times\ldots\times n_r\mathbb{Z}_r)^2$, then from (2.9*b*)

$$B(\xi,\beta)\equiv 0 \qquad (\mathrm{mod}\ n).$$

If $\gamma\in\Gamma_E(n_1,\ldots,n_r)$, then

$$I_{\gamma_k}=I\otimes\ldots\otimes h_k\otimes\ldots\otimes I$$

$$I_{\delta_k}=I\otimes\ldots\otimes g_k\otimes\ldots\otimes I.$$

Thus $I_{\gamma\xi}=I_\xi$ for all ξ.

The fact that Γ_E normalises the action of $\mathbb{Z}^r\times\mathbb{Z}^r$ follows from the combined action of $\gamma\in\Gamma_E$ and $\alpha\in(\mathbb{Z}^r\times\mathbb{Z}^r)$: for arbitrary w_ξ we have

$$w_\xi\overset{\gamma}{\mapsto}w_{\gamma^{-1}\xi}\overset{\alpha}{\mapsto}\omega^{B(\gamma^{-1}\xi,\alpha)}w_{\gamma^{-1}\xi}\overset{\gamma}{\mapsto}\omega^{B(\gamma^{-1}\xi,\alpha)}w_{\gamma(\gamma^{-1}\xi)}=\omega^{B(\xi,\gamma\alpha)}w_\xi.$$

Thus the combined action is equivalent to the action of just $\gamma\alpha=\alpha'\in\mathbb{Z}^r\times\mathbb{Z}^r$. This being the definition of the semidirect product we conclude the symmetry group is $\Gamma_E\rtimes(\mathbb{Z}^r\times\mathbb{Z}^r)$.

To obtain the symmetries of Fan and Wu mentioned previously, we must consider one more construction. We choose any Heisenberg matrix I_α such that I_α^2 is a multiple of the identity. This is only possible if n is even, in which case $\alpha=(\alpha_1',\ldots,\alpha_r',\alpha_1'',\ldots,\alpha_r'')$ must be such that

$$\alpha_i',\alpha_i''\equiv 0(\mathrm{mod}\ n_i) \qquad \text{if } n_i \text{ is odd}$$

$$\alpha_i',\alpha_i''\equiv \tfrac{1}{2}n_i(\mathrm{mod}\ n_i) \qquad \text{if } n_i \text{ is even}$$

implying that $2\alpha\equiv(0,0)$ and $I_\alpha^2=\pm$identity. Following the construction in Richey and Tracy (1986), one can show that $Z[S]$ is invariant under the transformation

$$S\mapsto I_\alpha\otimes I_\alpha^{-1}S$$

(assuming that the number of rows and columns of the lattice is even). If $n=2$, this, along with theorem 3.1, give us the symmetries of Fan and Wu.

Using theorem 3.1, we now obtain an interesting case of the symmetries which reduces to the so-called 'weak graph duality' (Wegner 1971, 1974, Fan and Wu 1970, Fan 1972, Johnson *et al* 1973). This relates a high-temperature model (all weights almost equal) to a low-temperature one (one weight, S_{00}^{00}, dominant).

For any $\alpha\in G_n$, $\alpha=(\alpha_1',\ldots,\alpha_r',\alpha_1'',\ldots,\alpha_r'')=(\alpha',\alpha'')$, consider the transformation

$$\alpha=\begin{pmatrix}\alpha'\\ \alpha''\end{pmatrix}\to\begin{pmatrix}\alpha''\\ -\alpha'\end{pmatrix}.$$

The matrix $\gamma\in\Gamma_E$ representing this transformation is

$$\gamma=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix} \qquad I=r\times r \text{ identity matrix.}$$

The corresponding transformation on the weights can be described as follows. Using X symmetry, any weight S_{ij}^{kl} can be written as

$$S_{ij}^{kl}=S_{0,j-i}^{k-i,l-i}=S^{a,b}$$

$$a=k-i\in X$$

$$b=l-i\in X.$$

It is straightforward to confirm that

$$S_{ij}^{kl} = S^{a,b} = \sum_{\alpha\in\mathbb{Z}_{n_1}\times\ldots\times\mathbb{Z}_{n_r}} w_{(-a,\alpha)}\omega^{-(n/n_1)b_1\alpha_1\ldots-(n/n_r)b_r\alpha_r}$$

and the associated inversion formula

$$w_{(a,b)} = \frac{1}{n}\sum_{\beta\in\mathbb{Z}_{n_1}\times\ldots\times\mathbb{Z}_{n_r}} S^{-a,\beta}\omega^{(n/n_1)b_1\beta_1+\ldots+(n/n_r)b_r\beta_r}.$$

Letting $(\alpha', \alpha'')\to(\alpha'', -\alpha')$ induces

$$S^{a,b}\to\sum_\alpha w_{(\alpha,a)}\omega^{-(n/n_1)b_1\alpha_1-\ldots-(n/n_r)b_r\alpha_r} = \tilde{S}^{a,b}.$$

Using the inversion formula

$$\tilde{S}^{a,b} = \sum_\alpha\frac{1}{n}\sum_\beta S^{-\alpha,\beta}\omega^{(n/n_1)a_1\beta_1+\ldots+(n/n_r)a_r\beta_r-(n/n_1)b_1\alpha_1-\ldots-(n/n_r)b_r\alpha_r}$$

$$= \frac{1}{n}\sum_{(\alpha,\beta)\in G_n} S^{-\alpha,\beta}\omega^{B((a,b),(\alpha,\beta))}.$$

If we let $T\to\infty$, $S_{ij}^{kl}\to 1$ i, j, k, l; hence

$$\tilde{S}^{a,b}\to\frac{1}{n}\sum_{(\alpha,\beta)}\omega^{B((a,b),(\alpha,\beta))}$$

$$= n\delta_0^{a_1}\ldots\delta_0^{a_r}\delta_0^{b_1}\ldots\delta_0^{b_r}$$

$$= \begin{cases} n & \text{if } a=b=(0,\ldots,0)\\ 0 & \text{otherwise.}\end{cases}$$

Hence the Γ_E transformation $(\alpha', \alpha'')\to(\alpha'', -\alpha')$ sends a high-temperature model to a low-temperature one, while preserving the partition function.

4. Discussion of the case $X = \mathbb{Z}_n$

If $X = \mathbb{Z}_n = \mathbb{Z}_{n_1}\oplus\ldots\oplus\mathbb{Z}_{n_r}$, then we have $(n_i, n_j) = 1$, $i\neq j$, i.e. the primes are distinct. This case has been previously investigated (Richey and Tracy 1986) so we show that, as expected, the new formalism of the previous sections results in no new symmetries. The isomorphism $\mathbb{Z}_{n_1}\oplus\ldots\oplus\mathbb{Z}_{n_r}\to\mathbb{Z}_n$ is the Chinese remainder theorem. This theorem asserts that for any r-tuple $(\alpha_1,\ldots,\alpha_r)\in\mathbb{Z}_{n_1}\oplus\ldots\oplus\mathbb{Z}_{n_r}$, there is a unique $\bar{\alpha}$ (mod n) such that $\bar{\alpha}\equiv\alpha_i$ (mod n_i), $i=1,\ldots,r$.

Let $l\equiv(n/n_i)(\text{mod } n_i)$, $i=1,\ldots,r$. Then since $(n_i, n/n_i)=1$ we have $(n_i, l)=1$ which implies $(n, l)=1$. Moreover for any $\alpha, \beta\in G_n$ we have

$$B(\alpha,\beta)\equiv l\langle\bar{\alpha},\bar{\beta}\rangle(\text{mod } n) \tag{4.1}$$

where $\bar{\alpha}=(\bar{\alpha}', \bar{\alpha}'')$, $\bar{\beta}=(\bar{\beta}', \bar{\beta}'')$, $\bar{\alpha}\equiv\alpha_i'$ (mod n_i), etc. To show this is true mod n, we need only to demonstrate (4.1) holds mod n_i, $i=1,\ldots,r$:

$$B(\alpha,\beta)=\sum_{k=1}^r\frac{n}{n_k}(\alpha_k'\beta_k''-\alpha_k''\beta_k')$$

$$\equiv(n/n_i)(\alpha_i'\beta_i''-\alpha_i''\beta_i')(\text{mod } n_i) \qquad \text{since } (n_i, n_j)=1 \text{ for } i\neq j$$

$$\equiv l(\bar{\alpha}'\bar{\beta}''-\bar{\alpha}''\beta')(\text{mod } n_i).$$

Under the map $\xi\in G_n\mapsto\bar{\xi}\in\mathbb{Z}_n\times\mathbb{Z}_n$ we can identify w_ξ with $w_{\bar{\xi}}$. We first see that the symmetry $w_\xi\to\omega^{B(\xi,\beta)}w_\xi$, $\xi,\beta\in G_n=(\mathbb{Z}_{n_1}\oplus\ldots\oplus\mathbb{Z}_{n_r})^2$ is equivalent to $w_{\bar{\xi}}\mapsto\omega^{l\langle\bar{\xi},\bar{\beta}\rangle}w_{\bar{\xi}}$,

$\xi, \beta \in \mathbb{Z}_n \times \mathbb{Z}_n$. Since $(l, n) = 1$, ω^l is a primitive nth root of unity, and hence no new symmetries are obtained.

A more interesting situation arises when we consider the effect of changing generators in H_X. This corresponds to choosing some $\gamma \in \Gamma_E$. The action of γ on $(\mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r})^2$ induces our action on $\mathbb{Z}_n^2$ as follows:

$$\begin{array}{ccc} \alpha \in (\mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r})^2 & \xrightarrow{\;\gamma\;} & \gamma\alpha \in (\mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r})^2 \\ {\scriptstyle \mathrm{CRT}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{CRT}} \\ \bar{\alpha} \in \mathbb{Z}_n^2 & \xrightarrow{\;A\;} & \overline{\gamma\alpha} \in \mathbb{Z}_n^2. \end{array}$$

Since the CRT and γ are linear, the map $\bar{\alpha} \mapsto \overline{\gamma\alpha}$ is also linear and hence has a matrix representation $\bar{\alpha} \mapsto \overline{\gamma\alpha} = A\bar{\alpha}$. We will show that $A \in \mathrm{SL}(2, \mathbb{Z}_n)$. This will suffice to show that the action $w_{\bar{\alpha}} \mapsto w_{\overline{\gamma\alpha}} = w_{A\bar{\alpha}}$ induces no new symmetries not discussed in Richey and Tracy (1986). Since $A\colon \mathbb{Z}_n^2 \to \mathbb{Z}_n^2$, it is enough to show that $\det(A) \equiv 1 \pmod n$. Now $\gamma \in \Gamma_E$ if and only if $B(\gamma\alpha, \gamma\beta) = B(\alpha, \beta)$ for all $\alpha, \beta \in G_n$ and $A \in \mathrm{SL}(2, \mathbb{Z}_n)$ if and only if $\langle A\bar{\alpha}, A\bar{\beta}\rangle \equiv \langle \bar{\alpha}, \bar{\beta}\rangle$ for all $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}_n^2$. In particular, for $\bar{\alpha} = (1, 0)$ and $\bar{\beta} = (0, 1)$

$$\langle A\bar{\alpha}, A\bar{\beta}\rangle = \det A$$

but by (4.1)

$$\begin{aligned} l\langle A\bar{\alpha}, B\bar{\beta}\rangle &= l\langle \overline{\gamma\alpha}, \overline{\gamma\beta}\rangle = B(\gamma\alpha, \gamma\beta) \\ &= B(\alpha, \beta) = \sum_{k=1}^{r} \frac{n}{n_k} \end{aligned}$$

since

$$\alpha = (\alpha'_1, \ldots, \alpha'_r, \alpha''_1, \ldots, \alpha''_r)$$

with $\alpha'_i = 1$, $\alpha''_i = 0$ and $\beta = (\beta'_1, \ldots, \beta'_r, \beta''_1, \ldots, \beta''_r)$ with $\beta'_i = 0$, $\beta''_i = 1$. Hence we have

$$l \det A = \sum_{k=1}^{r} \frac{n}{n_k}.$$

Consider this equation mod n_i and recall that $l \equiv (n/n_i) \pmod{n_i}$ and $(n_i, n_j) = 1$, $i \neq j$, we have

$$\det A = 1 \pmod{n_i} \qquad i = 1, \ldots, r$$

which implies $\det A = 1 \pmod n$.

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